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A NOTE ON SOME REPRESENTATIONS OF APPELL AND HORN FUNCTION

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Abstract. In the present note, we obtain certain new representations of the four Appell functions F_1, F_2, F_3 and F_4, and of the four Horn functions H_1, H_2, H_3 and H_4 in terms of series of generalized hypergeometric functions. By specializing certain parameters, some new interesting connections with generalized hypergeometric function and formulas are also obtained as special cases of our main results.

1. Introduction

There are many applications of Appell functions in various physical and chemical fields such as the radiation field problems [12], Hubbell rectangular source and its generalization [6], Non-relativistic theory [7], and hydrogen dipole matrix element [8]. Moreover, these series are used in quantum field theory, in particular, in the evaluation of Feynman integrals [16, 17]. Since 1985, Horn functions have been utilized as a fundamental concept in the compuatational sciences (as artificial intelligence and information processing) [5, 10, 11].

We start the present note with the expression of the generalized Gauss or the generalized hypergeometric series defined by [13]

$$ {}_pF_q\left(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, \tag{1.1}$$

whree $(\gamma)_n$ is the Pochhammer symbol or the factorial function defined by

$$ (\gamma)_n=\begin{cases}\gamma(\gamma+1)\cdots(\gamma+n-1), & n\in\mathbb{N}^*=1,2,3,\ldots;\gamma\in\mathbb{C}\\ 1, & n=0;\gamma\in\mathbb{C}\setminus\{0\}\end{cases}, \tag{1.2}$$

which can be expressed in terms of Gamma function as (see [18, p. 2 and p. 5])

$$ (\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)},\quad \gamma\neq 0,-1,-2,\ldots \tag{1.3}$$

Note that the series given by (1.1) converges for:

$$\begin{cases}|z|<\infty, & if\ p\le q,\\ |z|<1, & if\ p=q+1.\end{cases}$$

In view of (1.1), we consider here, the four Appel functions of two variables, which are defined below [19, 20]

$$ F_1\left[a,b,b';c;x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.4}$$

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with $\max\{|x|,\ |y|\} < 1$;

$$F_2\left[a, b, b'; c, c'; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.5}$$

with $|x| + |y| < 1$;

$$F_3\left[a, a', b, b'; c; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.6}$$

with $\max\{|x|,\ |y|\} < 1$;
and

$$F_4\left[a, b; c, c'; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.7}$$

with $\sqrt{|x|} + \sqrt{|y|} < 1$.

Following the investigation of Appell and Kampé de Fériet [1, p. 143], Horn defined ten hypergeometric functions of two variables. In the present work, we are interested in the following four Horn functions given by (see [19, 20])

$$H_1\left[\alpha, \beta, \gamma; \delta; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.8}$$

with $|x| < r,\ |y| < s,\ 4rs = (s-1)^2$;

$$H_2\left[\alpha, \beta, \gamma, \delta; \varepsilon; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\varepsilon)_m} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.9}$$

with $|x| < r,\ |y| < s,\ (r+1)s = 1$;

$$H_3\left[\alpha, \beta; \gamma; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.10}$$

with $|x| < r,\ |y| < s,\ r + (s - 1/2)^2 = 1/4$;
and

$$H_4\left[\alpha, \beta; \gamma, \delta; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_m(\delta)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.11}$$

with $|x| < r,\ |y| < s,\ 4r = (s-1)^2$.

For the above definitions and for the rest of the text, the denominator parameters are neither zero or a negative integer.

Recently, much work have been done on the representation of Humbert's hypergeometric functions in a series of Gauss's function ${}_2F_1$ [3, 4, 9, 14, 15]. These papers are the pioneer of the present note. The aim of this note is to find some representations of Appell and Horn functions in series of some generalized hypergeometric functions and to give some special cases of our main results.

2. Main results

Theorem 2.1. *For $c \neq 0, -1, -2, \ldots$, $|x| \neq 0$ and $|\frac{y}{x}| < 1$, the undermentioned result holds true:*

$$F_1\left[a, b, b'; c; x, y\right] = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} \frac{x^m}{m!} \, {}_2F_1\left(-m, b'; 1 - b - m; \frac{y}{x}\right). \tag{2.1}$$

Proof. Using (1.4) and by denoting the left-hand side of (2.1) by S_1 and with the help of the following identity (see [20])

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A(n,m) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} A(n,m-n), \tag{2.2}$$

we get

$$S_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(a)_m (b)_{m-n} (b')_n}{(c)_m} \frac{x^{m-n}}{(m-n)!} \frac{y^n}{n!}. \tag{2.3}$$

By using the identities (see [20])

$$(m-n)! = \frac{(-1)^n m!}{(-m)_n}, \tag{2.4}$$

and

$$(b)_{m-n} = \frac{(-1)^n (b)_m}{(1-b-m)_n}, \tag{2.5}$$

S_1 becomes

$$S_1 = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} \sum_{n=0}^{m} \frac{(-m)_n (b')_n}{(1-b-m)_n} \frac{(y/x)^n}{n!}. \tag{2.6}$$

The second summation over n in (2.6) is the hypergeometric function

$${}_2F_1\left(-m, b'; 1-b-m; \frac{y}{x}\right) \text{ (also see (1.1)),}$$

in view of which, we easily arrive at the result (2.1). □

Theorem 2.2. *For $c \neq 0, -1, -2, \ldots$, $|x| \neq 0$ and $|\frac{y}{x}| < 1$, the undermentioned transformation holds true:*

$$\begin{aligned} &F_2[a, b, b'; c, c'; x, y] \\ &= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} {}_3F_2\left(-m, 1-c-m, b'; 1-b-m, c'; \frac{-y}{x}\right). \end{aligned} \tag{2.7}$$

Proof. Using (2.2), (2.4) and (2.5) and by denoting the left-hand side of (2.7) by S_2, we get, after some simplifications, the following expression

$$S_2 = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} \sum_{n=0}^{m} \frac{(-m)_n (1-c-m)_n (b')_n}{(1-b-m)_n (c')_n} \frac{(-y/x)^n}{n!}. \tag{2.8}$$

The second summation over n is

$${}_3F_2\left(-m, 1-c-m, b'; 1-b-m, c'; \frac{-y}{x}\right),$$

and in view of that we complete the proof of Theorem 2.2. □

Theorem 2.3. *For $|x| \neq 0$ and $|\frac{y}{x}| < 1$, the undermentioned transformation holds true:*

$$\begin{aligned} &F_3[a, a', b, b'; c; x, y] \\ &= \sum_{m=0}^{\infty} (a)_m (b)_m \frac{x^m}{m!} {}_3F_3\left(-m, a', b'; 1-a-m, 1-b-m, c; \frac{-y}{x}\right). \end{aligned} \tag{2.9}$$

Proof. In this case, we consider the left-hand side of this last equation as S_3 and by using the same identities as in the above proofs, one finds

$$S_3 = \sum_{m=0}^{\infty} (a)_m (b)_m \frac{x^m}{m!} \sum_{n=0}^{m} \frac{(-m)_n (a')_n ((b')_n}{(1-a-m)_n (1-b-m)_n (c)_n} \frac{(-y/x)^n}{n!}. \tag{2.10}$$

The second summation in the right-hand side of S_3 is

$$ {}_3F_2\left(-m, a', b'; 1-a-m, 1-b-m, c; \frac{-y}{x}\right), $$

in view of which we get the required transformation (2.9) of Theorem 2.3. □

Theorem 2.4. *For $c \neq 0, -1, -2, \ldots$, $|x| \neq 0$ and $|\frac{y}{x}| < 1$, the undermentioned transformation holds true:*

$$ F_4\left[a, b; c, c'; x, y\right] = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}\, {}_2F_1\left(-m, 1-c-m; c'; \frac{y}{x}\right). \tag{2.11} $$

Proof. Follwing the proof of the above theorem, and by using the identities in (2.2), (2.4) and (2.5), the summation S_4, which involves F_4 given by (1.7), becomes

$$ S_4 = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} \sum_{n=0}^{m} \frac{(-m)_n (1-c-m)_n}{(c')_n} \frac{(y/x)^n}{n!}. \tag{2.12} $$

By using (1.1), one sees that this last summation is equal to

$$ {}_2F_1\left(-m, 1-c-m; c'; \frac{y}{x}\right), $$

and finally Theorem 2.4 is proved. □

Theorem 2.5. *Let $\delta \neq 0, -1, -2, \ldots$, $|x| \neq 0$, $|x| < r$, $|y| < s$ and $4rs = (s-1)^2$, then the undermentioned transformation holds:*

$$ \begin{aligned} H_1\left[\alpha, \beta, \gamma; \delta; x, y\right] &= \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\delta)_m} \frac{x^m}{m!} \\ &\times {}_3F_2\left(-m, \gamma, 1-\delta-m; \tfrac{1-(\alpha+m)}{2}, 1-\tfrac{(\alpha+m)}{2}; \tfrac{y}{4x}\right). \end{aligned} \tag{2.13} $$

Proof. By using (1.8), the identities (2.2), (2.4) and (2.5) and the following elementary identity

$$ (\lambda)_{2n} = 4^n \left(\frac{\lambda}{2}\right)_n \left(\frac{\lambda+1}{2}\right)_n, \tag{2.14} $$

the expression of H_1 becomes

$$ \begin{aligned} H_1\left[\alpha, \beta, \gamma; \delta; x, y\right] &= \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\delta)_m} \frac{x^m}{m!} \\ &\times \sum_{n=0}^{m} \tfrac{(-m)_n (\gamma)_n (1-\delta-m)_n}{(\frac{1-(\alpha+m)}{2})_n (1-\frac{(\alpha+m)}{2})_n} \tfrac{(y/4x)^n}{n!}. \end{aligned} \tag{2.15} $$

The second summation of this last equation is the hypergeometric function

$$ {}_3F_2\left(-m, \gamma, 1-\delta-m; \frac{1-(\alpha+m)}{2}, 1-\frac{(\alpha+m)}{2}; \frac{y}{4x}\right), $$

in view of which we complete the proof of Theorem 2.5. □

Theorem 2.6. *Let $\varepsilon \neq 0, -1, -2, \ldots$, $|x| \neq 0$, $|x| < r$, $|y| < s$, and $(r+1)s = 1$. Then*

$$ \begin{aligned} H_2\left[\alpha, \beta, \gamma, \delta; \varepsilon; x, y\right] &= \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\varepsilon)_m} \frac{x^m}{m!} \\ &\times {}_4F_3\left(-m, \gamma, \delta, 1-\varepsilon-m; \tfrac{1-(\alpha+m)}{2}, 1-\tfrac{(\alpha+m)}{2}, 1-\beta-m; \tfrac{-y}{4x}\right). \end{aligned} \tag{2.16} $$

Proof. The use of (1.9), and the identities (2.2), (2.4), (2.5) and (2.14) yields the following expression of H_2

$$H_2\left[\alpha,\beta,\gamma,\delta;\varepsilon;x,y\right]=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\varepsilon)_m}\frac{x^m}{m!} \times\sum_{n=0}^{m}\frac{(-m)_n(\gamma)_n(\delta)_n(1-\varepsilon-m)_n}{(\frac{1-(\alpha+m)}{2})_n(1-\frac{(\alpha+m)}{2})_n(1-\beta-m)_n}\frac{(-y/4x)^n}{n!}. \tag{2.17}$$

This completes the proof of Theorem 2.6 by noting that this last summation of (2.17) is equal to

$$_4F_3\left(-m,\gamma,\delta,1-\varepsilon-m;\frac{1-(\alpha+m)}{2},1-\frac{(\alpha+m)}{2},1-\beta-m;\frac{-y}{4x}\right).$$

□

Theorem 2.7. *let $\gamma\neq 0,-1,-2,\ldots$, $|x|\neq 0$, $|x|<r$, $|y|<s$, and $r+(s-1/2)^2=1/4$. Then*

$$H_3\left[\alpha,\beta;\gamma;x,y\right]=\sum_{m=0}^{\infty}\frac{(\frac{\alpha}{2})_m(\frac{\alpha+1}{2})_m}{(\gamma)_m}\frac{(4x)^m}{m!}\,{}_2F_1\left(-m,\beta;1-\alpha-2m;\frac{y}{x}\right). \tag{2.18}$$

Proof. By making use of (1.10) and the identities (2.2), (2.4), (2.5) and (2.14), the third Horn function can be expressed as

$$H_3\left[\alpha,\beta;\gamma;x,y\right]=\sum_{m=0}^{\infty}\frac{(\frac{\alpha}{2})_m(\frac{\alpha+1}{2})_m}{(\gamma)_m}\frac{(4x)^m}{m!}\sum_{n=0}^{m}\frac{(-m)_n(\beta)_n}{(1-\alpha-2m)_n}\frac{(y/x)^n}{n!}. \tag{2.19}$$

Since the second summation of the above expression is equal to

$$_2F_1\left(-m,\beta;1-\alpha-2m;\frac{y}{x}\right),$$

we get the desired result (2.18) of Theorem 2.7 directly follows. □

Theorem 2.8. *Let γ and $\delta\neq 0,-1,-2,\ldots$, $|x|\neq 0$, $|x|<r$, $|y|<s$, $4r=(s-1)^2$. Then*

$$H_4\left[\alpha,\beta;\gamma,\delta;x,y\right]=\sum_{m=0}^{\infty}\frac{(\frac{\alpha}{2})_m(\frac{\alpha+1}{2})_m}{(\gamma)_m}\frac{(4x)^m}{m!} \times{}_3F_2\left(-m,\beta,1-\gamma-m;\delta,1-\alpha-2m;\tfrac{-y}{x}\right), \tag{2.20}$$

Proof. As in the above proofs, (1.11) and the identities (2.2), (2.4), (2.5) and (2.14), yield the expression of H_4 as follows

$$H_4\left[\alpha,\beta;\gamma,\delta;x,y\right]=\sum_{m=0}^{\infty}\frac{(\frac{\alpha}{2})_m(\frac{\alpha+1}{2})_m}{(\gamma)_m}\frac{(4x)^m}{m!} \times\sum_{n=0}^{m}\frac{(-m)_n(\beta)_n(1-\gamma-m)_n}{(\delta)_n(1-\alpha-2m)_n}\frac{(-y/x)^n}{n!}. \tag{2.21}$$

The definition (1.1) of the hypergeometric function gives that the last summation of (2.21) is equal to

$$_3F_2\left(-m,\beta,1-\gamma-m;\delta,1-\alpha-2m;\frac{-y}{x}\right).$$

This completes the proof of Theorem 2.8. □

3. Special cases

Corollary 3.1. *Let* $c \neq 0, -1, -2, \ldots$, $|x| \neq 0$ *and* $|x| < 1$. *Then*

$$F_1\left[a, b; b'; c; x, x\right] = {}_2F_1\left(a, b + b'; c; x\right). \tag{3.1}$$

Proof. Taking $y = x$ in (2.1), we can write

$$F_1\left[a, b, b'; c; x, x\right] = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} \, {}_2F_1\left(-m, b'; 1 - b - m; 1\right). \tag{3.2}$$

Now, by using the identity (see [13, p. 489])

$$ {}_2F_1\left(-m, b_1; c_1; 1\right) = \frac{(c_1 - b_1)_m}{(c_1)_m}, \tag{3.3}$$

and

$$\Gamma(\alpha - n) = \frac{(-1)^n}{(1 - \alpha)_n} \Gamma(\alpha), \tag{3.4}$$

(3.2) becomes

$$F_1\left[a, b, b'; c; x, x\right] = \sum_{m=0}^{\infty} \frac{(a)_m (b + b')_m}{(c)_m} \frac{x^m}{m!}, \tag{3.5}$$

and Corollary 3.1 is proved and is identical to (61) of [13, p. 452]. □

Corollary 3.2. *Let* c *and* $c' \neq 0, -1, -2, \ldots$, $c + c' \neq 1$, $|x| \neq 0$ *and* $|x| < 1$. *Then*

$$F_4\left[a, b; c, c'; x, x\right] = {}_4F_3\left(a, b, \frac{c + c'}{2}, \frac{c + c' - 1}{2}; c, c', c + c' - 1; 4x\right). \tag{3.6}$$

Proof. We start from the identity (3.3) with $b_1 = 1 - c - m$ and $c_1 = c'$, so that

$$ {}_2F_1\left(-m, 1 - c - m; c'; 1\right) = \frac{(c' - 1 + c + m)_m}{(c')_m}. \tag{3.7}$$

Since with the help of the following relations

$$(\alpha + m)_m = \frac{(\alpha)_{2m}}{(\alpha)_m} \tag{3.8}$$

and

$$(\alpha)_{2m} = 4^m \left(\frac{\alpha}{2}\right)_m \left(\frac{\alpha + 1}{2}\right)_m, \tag{3.9}$$

we have

$$(\alpha + m)_m = \frac{4^m \left(\frac{\alpha}{2}\right)_m \left(\frac{\alpha+1}{2}\right)_m}{(\alpha)_m}, \tag{3.10}$$

so that, we can write

$$ {}_2F_1\left(-m, 1 - c - m; c'; 1\right) = \frac{4^m \left(\frac{c+c'-1}{2}\right)_m \left(\frac{c+c'}{2}\right)_m}{(c + c' - 1)_m (c')_m}. \tag{3.11}$$

Finally from Theorem 2.4 we deduce that

$$F_4\left[a, b; c, c'; x, x\right] = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{\left(\frac{c+c'-1}{2}\right)_m \left(\frac{c+c'}{2}\right)_m}{(c')_m (c + c' - 1)_m} \frac{(4x)^m}{m!} \tag{3.12}$$

This completes the proof of Corollary 3.2. □

Corollary 3.3. *For $|x| \neq 0$, and $|\frac{y}{x}| < 1$, we have*

$$\sum_{m=0}^{\infty} (b)_m \frac{x^m}{m!} \, {}_2F_1\left(-m, b'; 1-b-m; \frac{y}{x}\right) = \frac{1}{(1-x)^b (1-y)^{b'}}. \tag{3.13}$$

This equation can be rearranged in the case of $x = y$, as

$$\sum_{m=0}^{\infty} (b)_m \frac{x^m}{m!} {}_2F_1\left(-m, b'; 1-b-m; 1\right) = \frac{1}{(1-x)^{(b+b')}}. \tag{3.14}$$

Proof. With the help of the following identity (see [19])

$$F_1\left[a, b, b'; a; x, y\right] = (1-x)^{-b}(1-y)^{-b'}, \tag{3.15}$$

and by taking $a = c$ in (2.1), the above equation becomes (3.13). This proves the Corollary 3.3. □

Corollary 3.4. *For $c \neq 0, -1, -2, \ldots$, $|x| \neq 0$, and $\Re(c - a - b - b') > 0$, the undermentioned results hold true:*

$$\sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} \, {}_2F_1\left(-m, b'; 1-b-m; 1\right) = {}_2F_1\left(a, b+b'; c; x\right) \tag{3.16}$$

and

$$\sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m (1)_m} \, {}_2F_1\left(-m, b'; 1-b-m; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b-b')}{\Gamma(c-a)\Gamma(c-b-b')}. \tag{3.17}$$

Proof. Eqs. (3.16) and (3.17) are proved easily from Thoerem 2.1 and by using the following identities [19]

$$F_1\left[a, b, b'; a; x, x\right] = {}_2F_1\left(a, b+b'; c; x\right), \tag{3.18}$$

and

$$F_1\left[a, b, b'; c; 1, 1\right] = \frac{\Gamma(c)\Gamma(c-a-b-b')}{\Gamma(c-a)\Gamma(c-b-b')}. \tag{3.19}$$

Taking $c = b$ and $c' = b'$ in Thoerem 2.2, and using the following identity (see [19])

$$F_2\left[a, b, b'; b, b'; x, y\right] = (1-x-y)^{-a}, \tag{3.20}$$

and the following elementary identity (see [13, p. 453])

$$ {}_1F_0\left(a; z\right) = (1-z)^{-a}, \tag{3.21}$$

it's easy to prove the following Corollary: □

Corollary 3.5. *For $|x| \neq 0$, and $|\frac{y}{x}| < 1$, the undermentioned transformation holds true:*

$$\sum_{m=0}^{\infty} (a)_m \left(1 + \frac{x}{y}\right)^m \frac{(x)^m}{m!} = (1-x-y)^{-a}. \tag{3.22}$$

Not that if $x = y$, this equation becomes:

$$\sum_{m=0}^{\infty} (a)_m \frac{(2x)^m}{m!} = (1-2x)^{-a}. \tag{3.23}$$

Corollary 3.6. *For $|x| \neq 0$, and $|\frac{y}{x}| < 1$, the undermentioned transformation holds true:*

(3.24)
$$\sum_{m=0}^{\infty} (b)_m \frac{x^m}{m!} \, {}_3F_2\left(-m, 1-a-m, b'; 1-b-m, b; \frac{-y}{x}\right) = (1-y)^{b-a}(1-x-y)^{-b}.$$

Proof. The result in (3.24) can be deduced from Theorem 2.2 with the use of the following identity (see [19])

$$F_2\left[a,b,b';a,b';x,x\right]=(1-y)^{b-a}(1-x-y)^{-b}. \tag{3.25}$$

Note that if we take $x=y$, (3.24) becomes

$$\sum_{m=0}^{\infty}(b)_m\frac{x^m}{m!}\,{}_3F_2\left(-m,1-a-m,b';1-b-m,b;-1\right)=(1-x)^{b-a}(1-2x)^{-b}. \tag{3.26}$$

Using the following identity (see [19])

$$F_2\left[a,b,b';b,a;x,y\right]=(1-x)^{b'-a}(1-x-y)^{-b'}, \tag{3.27}$$

and by taking $c=b$ and $c'=a$ in (2.6), we prove the following Corollary: □

Corollary 3.7. *For $|x|\neq 0$, and $|\frac{y}{x}|<1$, the undermentioned transformation holds true:*

$$\sum_{m=0}^{\infty}(a)_m\frac{x^m}{m!}\,{}_2F_1\left(-m,b';a;\frac{-y}{x}\right)=(1-x)^{b'-a}(1-x-y)^{-b'}, \tag{3.28}$$

which becomes, if $x=y$, the elementary identity

$$\sum_{m=0}^{\infty}(a-b')_m\frac{x^m}{m!}=(1-x)^{b'-a}(1-2x)^{-b'}. \tag{3.29}$$

This last equation is obtained also by using the identity (see [19])

$${}_2F_1\left(-n,b;c;1\right)=\frac{(c-b)_n}{(c)_n}. \tag{3.30}$$

For the thirth Appell function, and by considering that $|x|\neq 0$, and $|\frac{y}{x}|<1$, we use the identity (see [13])

$$\begin{aligned}F_3\left[a,a',1,1;a+a',x,y\right]&=(x+y-xy)^{-1}\\&\times\{x\,{}_2F_1\left(a,1;a+a';x\right)+y\,{}_2F_1\left(a',1;a+a';y\right)\},\end{aligned} \tag{3.31}$$

and by taking in Theorem 2.3, $b=b'=1$ and $c=a+a'$, one proves the following Corollary:

Corollary 3.8. *For $|x|\neq 0$, and $|\frac{y}{x}|<1$, the undermentioned transformation holds true:*

$$\begin{aligned}&\sum_{m=0}^{\infty}(a)_m x^m\,{}_2F_2\left(a',1;1-a-m,a+a';\frac{-y}{x}\right)\\&=(x+y-xy)^{-1}\left\{x\,{}_2F_1\left(a,1;a+a';x\right)+y\,{}_2F_1\left(a',1;a+a';y\right)\right\}.\end{aligned} \tag{3.32}$$

This equation becomes, if we take $x=y$

$$\begin{aligned}&\sum_{m=0}^{\infty}(a)_m\,x^m\,{}_2F_2\left(a',1;1-a-m,a+a';-1\right)\\&=\tfrac{1}{2-x}\{{}_2F_1\left(a,1;a+a';x\right)+{}_2F_1\left(a',1;a+a';x\right)\},\end{aligned} \tag{3.33}$$

which becomes with $a=a'$

$$\sum_{m=0}^{\infty}(a)_m\,x^m\,{}_2F_2\left(a,1;1-a-m,2a;-1\right)=\frac{2}{2-x}\,{}_2F_1\left(a,1;2a;x\right). \tag{3.34}$$

Corollary 3.9. *For $|x|\neq 0$ and $c,(a+b)\neq 0,-1,-2,\ldots$, the undermentioned transformation holds true:*

$$\begin{aligned}&\sum_{m=0}^{\infty}(a)_m(b)_m\frac{x^m}{m!}\,{}_3F_3\left(-m,a,b;1-a-m,1-b-m,c;-1\right)\\&={}_4F_3\left(a,b,\tfrac{a+b}{2},\tfrac{a+b+1}{2};a+b,\tfrac{c}{2},\tfrac{c+1}{2};x^2\right).\end{aligned} \tag{3.35}$$

Proof. From Bailay's formula (see [2])

$$F_3\left[a, a, b, b; c; x, -x\right] = {}_4F_3\left(a, b, \tfrac{a+b}{2}, \tfrac{a+b+1}{2}; a+b, \tfrac{c}{2}, \tfrac{c+1}{2}; x^2\right), \tag{3.36}$$

with $|x| \neq 0$, and by putting $a = a'$, $b = b'$ and $y = -x$ in Theorem 2.3, the required proof easily follows. □

Corollary 3.10. *For $c, c', c+c'-1 \neq 0, -1, -2, \ldots,$ and $|x| \neq 0$, the undermentioned transformation holds true:*

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} \, {}_2F_1\left(-m, 1-c-m; c'; 1\right) \\ &= {}_4F_3\left(a, b, \tfrac{c+c'}{2}, \tfrac{c+c'-1}{2}; c, c', c+c'-1; 4x\right). \end{aligned} \tag{3.37}$$

Proof. With the help of Theorem 2.4 by taking $x = y$ and by using the following Buchnall's formula (see [19])

$$F_4\left[a, b; c, c', x, x\right] = {}_4F_3\left(a, b, \frac{c+c'}{2}, \frac{c+c'-1}{2}; c, c', c+c'-1; 4x\right), \tag{3.38}$$

one finds (3.37).
Now, by taking $x = -y$ and $c' = c$ in Theorem 2.4, and using the Srivastava's formula [19]

$$F_4\left[a, b; c, c, x, -x\right] = {}_4F_3\left(\frac{a}{2}, \frac{a+1}{2}, \frac{b}{2}, \frac{b+1}{2}; c, \frac{c}{2}, \frac{c+1}{2}; -4x^2\right), \tag{3.39}$$

one can prove easly the following Corollary: □

Corollary 3.11. *For $c \neq 0, -1, -2, \ldots,$ and $|x| \neq 0$, the undermentioned result holds:*

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} \, {}_2F_1\left(-m, 1-c-m; c; -1\right) \\ &= {}_4F_3\left(\tfrac{a}{2}, \tfrac{a+1}{2}, \tfrac{b}{2}, \tfrac{b+1}{2}; c, \tfrac{c}{2}, \tfrac{c+1}{2}; -4x^2\right), \end{aligned} \tag{3.40}$$

Next, we use the identity (see [1])

$$\begin{aligned} F_4\left[a, a+\tfrac{1}{2}; c, \tfrac{1}{2}, x, y\right] &= \tfrac{1}{2}(1+\sqrt{y})^{-2a} \, {}_2F_1\left(a, a+\tfrac{1}{2}; c; \tfrac{x}{(1+\sqrt{y})^2}\right) \\ &+ \tfrac{1}{2}(1-\sqrt{y})^{-2a} \, {}_2F_1\left(a, a+\tfrac{1}{2}; c; \tfrac{x}{(1-\sqrt{y})^2}\right) \end{aligned} \tag{3.41}$$

and by putting in Theorem 2.4, $b = a + \frac{1}{2}$ and $c' = \frac{1}{2}$, we arrive at the following Corollary:

Corollary 3.12. *The undermentioned result holds:*

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{(a)_m (a+\frac{1}{2})_m}{(c)_m} \frac{x^m}{m!} \, {}_2F_1\left(-m, 1-c-m; \frac{1}{2}; \frac{y}{x}\right) = \frac{1}{2}(1+\sqrt{y})^{-2a} \\ &\times {}_2F_1\left(a, a+\tfrac{1}{2}; c; \tfrac{x}{(1+\sqrt{y})^2}\right) + \tfrac{1}{2}(1-\sqrt{y})^{-2a} {}_2F_1\left(a, a+\tfrac{1}{2}; c; \tfrac{x}{(1-\sqrt{y})^2}\right). \end{aligned} \tag{3.42}$$

On setting $x = y$, (3.39) *becomes*

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{(a)_m (a+\frac{1}{2})_m}{(c)_m} \frac{x^m}{m!} \, {}_2F_1\left(-m, 1-c-m; \frac{1}{2}; 1\right) = \frac{1}{2}(1+\sqrt{x})^{-2a} \\ &\times {}_2F_1\left(a, a+\tfrac{1}{2}; c; \tfrac{x}{(1+\sqrt{x})^2}\right) + \tfrac{1}{2}(1-\sqrt{x})^{-2a} \, {}_2F_1\left(a, a+\tfrac{1}{2}; c; \tfrac{x}{(1-\sqrt{x})^2}\right). \end{aligned} \tag{3.43}$$

If one takes $b = c$ and $a = c'$ in Theorem 2.4, (2.10) *becomes*

$$F_4\left[a, b; b, a; X, Y\right] = \sum_{m=0}^{\infty} (a)_m \frac{x^m}{m!} \, {}_2F_1\left(-m, 1-b-m; a; \frac{Y}{X}\right). \tag{3.44}$$

If we put

$$X = \frac{-x}{(1-x)(1-y)}, \tag{3.45}$$

and

$$Y = \frac{-y}{(1-x)(1-y)}, \tag{3.46}$$

and by using the Bailey's formula (see [2])

$$F_4\left[\alpha, \beta; \beta, \alpha; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right] = \frac{(1-x)^{\alpha}(1-y)^{\beta}}{(1-xy)}, \tag{3.47}$$

we deduce the following Corollary:

Corollary 3.13. *The undermentioned transformation holds true:*

$$\sum_{m=0}^{\infty}(a)_m \frac{x^m}{m!}\, {}_2F_1\left(-m, 1-b-m; a; \frac{y}{x}\right) = \frac{(1-x)^{a}(1-y)^{b}}{(1-xy)}. \tag{3.48}$$

Now if we take $b = c'$, (3.44) *can be written as*

$$F_4\left[a, b; b, b; X, Y\right] = \sum_{m=0}^{\infty}(a)_m \frac{x^m}{m!}\, {}_2F_1\left(-m, 1-b-m; b; \frac{Y}{X}\right). \tag{3.49}$$

With the help of the following reduction formula of (see Bailey [2])

$$\begin{aligned} &F_4\left[\alpha, \beta; \beta, \beta; \tfrac{-x}{(1-x)(1-y)}, \tfrac{-y}{(1-x)(1-y)}\right] \\ &= (1-x)^{\alpha}(1-y)^{\alpha}\, {}_2F_1\left(\alpha, \alpha-\beta+1; \beta; xy\right), \end{aligned} \tag{3.50}$$

we derive the following Corollary:

Corollary 3.14. *For* $|x| \neq 0$, $|\frac{y}{x}| < 1$ *and* $b \neq 0, -1, -2, \ldots$, *the undermentioned transformation holds true:*

$$\begin{aligned} &\sum_{m=0}^{\infty}(a)_m \frac{x^m}{m!}\, {}_2F_1\left(-m, 1-b-m; b; \frac{y}{x}\right) \\ &= (1-x)^{a}(1-y)^{a}\, {}_2F_1\left(a, a-b+1; b; xy\right), \end{aligned} \tag{3.51}$$

This equation can be rewritten, in the case of $x = y$, *as*

$$\begin{aligned} &\sum_{m=0}^{\infty}(a)_m \frac{x^m}{m!}\, {}_2F_1\left(-m, 1-b-m; b; 1\right) \\ &= (1-x)^{2a}\, {}_2F_1\left(a, a-b+1; b; x^2\right). \end{aligned} \tag{3.52}$$

Finally, we apply the reduction formula of Bailey (see [2])

$$\begin{aligned} &F_4\left[a, b; a-b+1, b; \tfrac{-x}{(1-x)(1-y)}, \tfrac{-y}{(1-x)(1-y)}\right] \\ &\quad = (1-y)^{a}\, {}_2F_1\left(a, b; a-b+1; \tfrac{-x(1-y)}{(1-x)}\right). \end{aligned} \tag{3.53}$$

From (2.10), *we can write the relation*

$$F_4\left[a, b; a-b+1, b; X, Y\right] = \sum_{m=0}^{\infty}\frac{(a)_m (b)_m}{(c)_m}\frac{x^m}{m!}\, {}_2F_1\left(-m, b-a; b; \frac{Y}{X}\right). \tag{3.54}$$

Now, from (3.53) *and* (3.54)*,we deduce the following Corollary:*

Corollary 3.15. *The undermentioned transformation holds true:*

$$\begin{aligned} &\sum_{m=0}^{\infty}\frac{(a)_m (b)_m}{(c)_m}\frac{x^m}{m!}\, {}_2F_1\left(-m, b-a; b; \frac{y}{x}\right) \\ &\quad = (1-y)^{a}\, {}_2F_1\left(a, b; a-b+1; \tfrac{-x(1-y)}{(1-x)}\right), \end{aligned} \tag{3.55}$$

Note that if we put $x = y$ *in* (3.55), *we obtain*

$$\sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} \, {}_2F_1\left(-m, b-a; b; 1\right) = (1-x)^a \, {}_2F_1\left(a, b; a-b+1; -x\right). \tag{3.56}$$

4. Conclusion

We have developed in this note, some representations of Appell and Horn functions in terms of generalized Gauss functions ${}_2F_1$, ${}_3F_2$, ${}_3F_3$ and ${}_4F_3$. Some special cases are deduced from our main results. New aplications of these obtained series in laser fields, will be published soon.

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